Exercises
Computational Intelligence Lab
SS 2020

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(Sparse Coding and Wavelets)

Problem 1 (Signal denoising):

Let $x(t)$ be a function expressed as a weighted sum of K basis functions $u_1(t), \dots, u_K(t)$:

$$x(t) = \sum_{k=1}^{K} z_k u_k(t)$$

For the sake of simplicity, we will consider orthonormal basis functions (e.g. normalized Haar Wavelets) that take discrete values.Thus, we can express these basis functions as vectors:

$$\mathbf{x} = \sum_{k=1}^{K} z_k \mathbf{u}_k = \mathbf{U}\mathbf{z}$$

For a fixed basis, we want to find a good approximation for $\mathbf{x}$ using only $\tilde{K}$ coefficients, where $\tilde{K} < K$. We denote the approximation $\hat{\mathbf{x}}$:

$$\hat{\mathbf{x}} = \sum_{k=1}^{\tilde{K}} z_k \mathbf{u}_k$$

If we consider only orthonormal bases, we can formulate the compression problem as picking from the original coefficients $z_1, \dots, z_K$ a subset $\tilde{K}$ of them which minimize the approximation error.

Let σ be a permutation of indices $\{1, \dots, K\}$ and $\hat{\mathbf{x}}_\sigma$ the function that uses the coefficients corresponding to the first $\tilde{K}$ indices of the permutation σ :

$$\hat{\mathbf{x}}_\sigma = \sum_{k=1}^{\tilde{K}} z_{\sigma(k)} \mathbf{u}_{\sigma(k)}$$

Question: Find the permutation σ^{min} which minimizes the L^2 approximation error $\left\|\mathbf{x} - \hat{\mathbf{x}}_\sigma\right\|_2^2$:

$$\sigma^{min} = \underset{\sigma}{\operatorname{argmin}} \left\|\mathbf{x} - \hat{\mathbf{x}}_\sigma\right\|_2^2$$

Also keep in mind that orthonormal basis means that the vectors comprising the basis are mutually orthogonal (zero inner product) and have unit length:

$$\begin{aligned} \langle \mathbf{u}_k, \mathbf{u}_l \rangle &= 0, \qquad k \neq l \\ \langle \mathbf{u}_k, \mathbf{u}_k \rangle &= 1 \end{aligned}$$

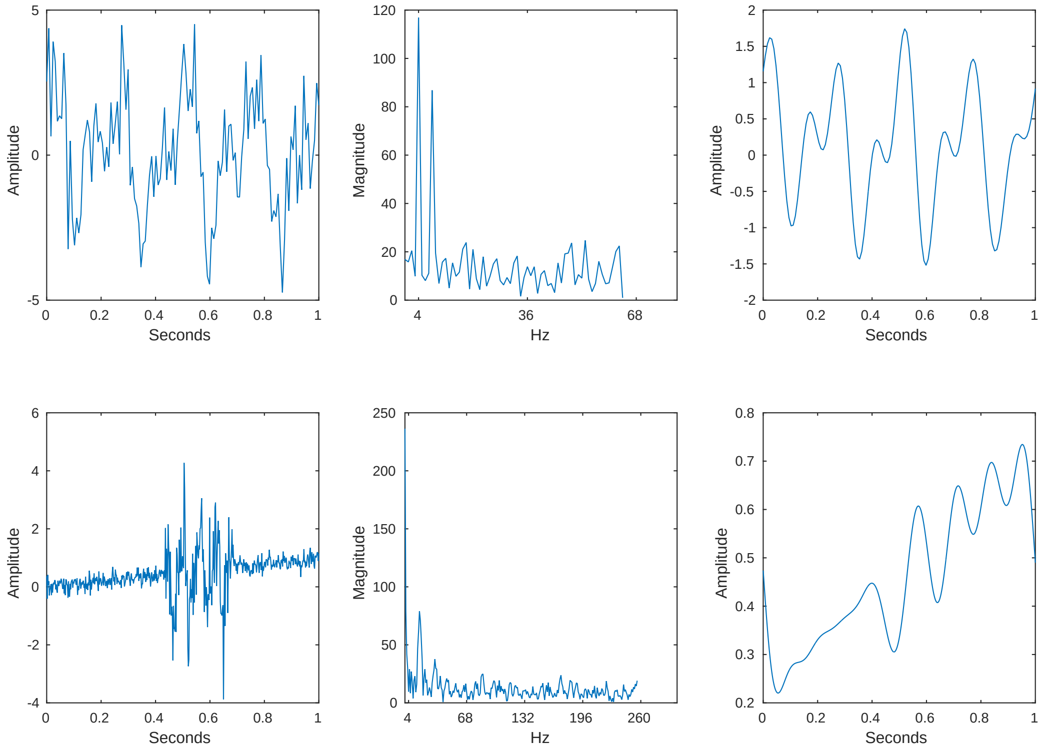
Problem 2 (1D signal compression and Haar wavelets):

You get now an opportunity to check your intuition from the last exercise using Haar wavelets. Please find the iPython notebook `ex2.ipynb` from

github.com/dalab/lecture_cil_public/tree/master/exercises/2020/ex10/ex2.ipynb, and answer the questions.

Problem 3 (Choice of dictionary is crucial):

The figure below shows two different 1-D signals (left column) with their corresponding spectrum obtained using the FFT (middle column). In the right column, we show a signal obtained by discarding part of the frequencies in the spectrum.



(i) Write down the formula to obtain the spectrum in the middle column of the previous figure, in terms of linear transformation or change of basis (assuming a given basis, i.e. the orthogonal matrix $\mathbf{U}$) applied to the original signal $\mathbf{x}$.

(ii) Write down the inverse formula to obtain the reconstructed signal in the right column in terms of linear transformation (change of basis) applied to the filtered spectrum $\hat{\mathbf{z}}$.

(iii) What part of the signal would you discard to obtain the reconstructed signal? Draw a rectangle on each spectrum in the middle column where everything inside the rectangle is kept for the reconstruction.

(iv) **true/false** The Wavelet transform is a better choice than Fourier for the first signal (top row).

(v) **true/false** The Wavelet transform is a better choice than Fourier for the second signal (bottom row).

(vi) Looking at the middle figure in the top row, what do the first peaks in the spectrum correspond to?

Problem 4 (Image compression):

Please find the iPython notebook `ex4.ipynb` from

github.com/dalab/lecture_cil_public/tree/master/exercises/2020/ex10/ex4.ipynb,

and answer the questions.