Exercises
Computational Intelligence Lab
SS 2020

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(Sparse Coding and Dictionary Learning)

Problem 1 (Sparse coding with overcomplete dictionary):

Given a signal $\mathbf{x} = (3, 1, -2) \in \mathbb{R}^3$ and an overcomplete dictionary $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4] \in \mathbb{R}^{3\times 4}$,

$$\mathbf{U} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},$$

find the sparse representation $\mathbf{z}$ of the signal $\mathbf{x}$ with $\|\mathbf{z}\|_0 \leq 2$.

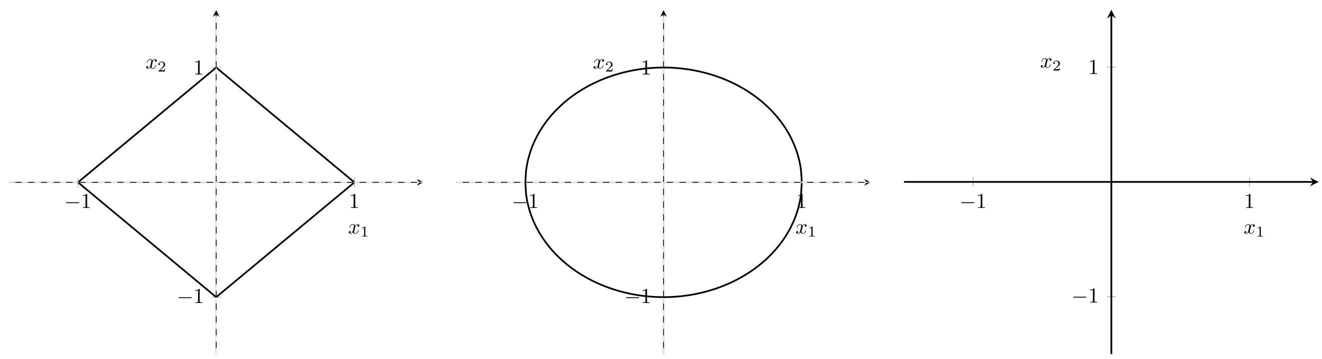
a. Find the atom $\mathbf{u}^{(1)}$ that minimizes the reconstruction error $\|\mathbf{x} - \hat{\mathbf{x}}^{(0)}\|_2^2$ where $\hat{\mathbf{x}}^{(0)} = z^{(1)}\mathbf{u}^{(1)}$, and compute the residual $\mathbf{r}^{(1)} = \mathbf{x} - \hat{\mathbf{x}}^{(0)}$.

b. Find the atom $\mathbf{u}^{(2)}$ that minimizes the reconstruction error $\|\mathbf{r}^{(1)} - \hat{\mathbf{x}}^{(1)}\|_2^2$ where $\hat{\mathbf{x}}^{(1)} = z^{(2)}\mathbf{u}^{(2)}$.

c. Write down the sparse representation $\mathbf{z}$ of signal $\mathbf{x}$.

Problem 2 (Compressed Sensing):

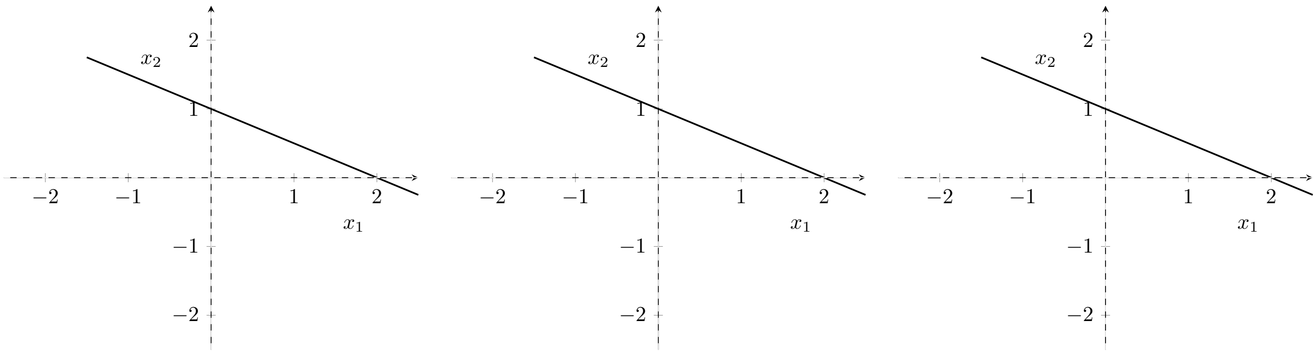
a. Map each of the three equations $\|\mathbf{x}\|_2 = 1$, $\|\mathbf{x}\|_1 = 1$, and $\|\mathbf{x}\|_0 = 1$ to a plot among a., b., or c. on the following figure. Note that $\mathbf{x}$ is a 2D vector with coordinates x_1 and x_2 (i.e. $\mathbf{x} = \left[x_1, x_2\right]$).



a. ....   b. ....   c. ....

b. Show the solution of each optimization problem on plots a., b., and c. of the following figure.

$\min \|\mathbf{x}\|_2$ Subject to $\frac{1}{2}x_1 + x_2 = 1$ | $\min \|\mathbf{x}\|_1$ Subject to $\frac{1}{2}x_1 + x_2 = 1$ | $\min \|\mathbf{x}\|_0$ Subject to $\frac{1}{2}x_1 + x_2 = 1$



a. b. c.

c. We can formulate the above three optimization problem as

$$\min \|\mathbf{x}\|_p$$
$$\text{subject to } \frac{1}{2}x_1 + x_2 = 1,$$

where $p \in \{0, 1, 2\}$. Mark the right sentence using your previous answers.

[ ] Solutions of the constrained problems have intersection for $p = 1$ and $p = 0$.

[ ] Solutions of the constrained problems have intersection for $p = 2$ and $p = 0$.

Problem 3 (Compressed Sensing):

Please find the iPython notebook `Compressed_sensing.ipynb` from

github.com/dalab/lecture_cil_public/tree/master/exercises/2020/ex11/ex3.ipynb

and answer the questions in this file.

Problem 4 (Matching Pursuit Algorithm):

In the last tutorial session, you have seen that the matching pursuit algorithm converges. In this exercise, you will show some limitations of the algorithm.

a. Find an overcomplete dictionary and a vector $\mathbf{x}$ such that the approximation $\mathbf{x}$ resulting from the matching pursuit algorithm will never exactly equal $\mathbf{x}$ no matter the number of iterations. Note that this implies that at least one atom will be selected multiple times.

Hint: Look for $\mathbf{U} \in \mathbb{R}^{2\times 3}$ such that MP keeps alternating between $\mathbf{U}_{\bullet 1}$ and $\mathbf{U}_{\bullet 2}$

b. Find an instance where the sparse representation returned by matching pursuit (assuming that after some number of iterations, the approximation is perfect) is not optimal, i.e. there is a different representation for which the 0-norm is strictly smaller.

Hint: Look for $\mathbf{U} \in \mathbb{R}^{2\times 3}$ and $\mathbf{x} \in \mathbb{R}^2$ such that you can find a representation $\mathbf{z}$ "by hand" with $||\mathbf{z}||_0 = 2$ but MP returns a representation $\mathbf{z}_{\text{MP}}$ with $||\mathbf{z}_{\text{MP}}||_0 = 3$