Exercises
Computational Intelligence Lab
SS 2020

Machine Learning Institute
Dept. of Computer Science, ETH Zürich
Prof. Dr. Thomas Hofmann
Web http://da.inf.ethz.ch/cil

Series 10, May 14-15, 2020 (Sparse Coding and Wavelets)

Problem 1 (Signal denoising):

$$\begin{aligned}
\|\mathbf{x} - \hat{\mathbf{x}}_\sigma\|_2^2 &= \langle \mathbf{x} - \hat{\mathbf{x}}_\sigma, \mathbf{x} - \hat{\mathbf{x}}_\sigma \rangle \\
&= \langle \sum_{k=1}^{K} z_k \mathbf{u}_k - \sum_{k=1}^{\tilde{K}} z_{\sigma(k)} \mathbf{u}_{\sigma(k)}, \sum_{l=1}^{K} z_l \mathbf{u}_l - \sum_{l=1}^{\tilde{K}} z_{\sigma(l)} \mathbf{u}_{\sigma(l)} \rangle \\
&= \langle \sum_{k=\tilde{K}+1}^{K} z_{\sigma(k)} \mathbf{u}_{\sigma(k)}, \sum_{l=\tilde{K}+1}^{K} z_{\sigma(l)} \mathbf{u}_{\sigma(l)} \rangle \\
&= \sum_{k=\tilde{K}+1}^{K} \sum_{l=\tilde{K}+1}^{K} z_{\sigma(k)} z_{\sigma(l)} \langle \mathbf{u}_{\sigma(k)}, \mathbf{u}_{\sigma(l)} \rangle \qquad \text{(because of the orthonormality of the basis)} \\
&= \sum_{k=\tilde{K}+1}^{K} (z_{\sigma(k)})^2
\end{aligned}$$

Thus, the optimal perturbation σ is obtained by:

$$\sigma^{\mathsf{min}} = \underset{\sigma}{\operatorname{argmin}} \left\{ \sum_{k=\hat{K}+1}^{K} (z_{\sigma(k)})^2 \right\}$$

It is obvious that the above expression is minimized when the permutation sorts the *noisy* coefficients z in order of decreasing magnitude.

Problem 2 (1D signal compression and Haar wavelets):

Please the solution at `ex2-sol.ipynb` from

github.com/dalab/lecture_cil_public/tree/master/exercises/ex10/ex2-sol.ipynb.

Problem 3 (Choice of dictionary is crucial):

The figure above shows two different 1-D signals (left column) with their corresponding spectrum obtained using the FFT (middle column). In the right column, we show a signal obtained by discarding part of the frequencies in the spectrum.

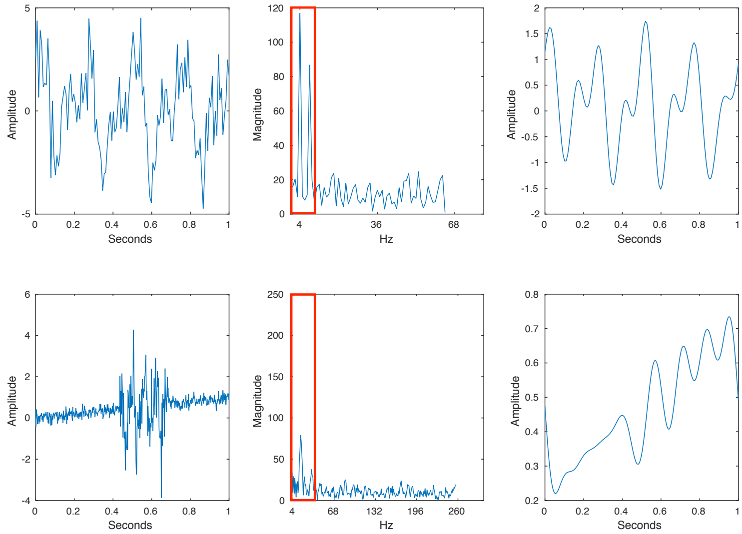
(i) Write down the formula to obtain the spectrum in the middle column of the previous figure, in terms of linear transformation or change of basis (assuming a given basis $\mathbf{U}$) applied to the original signal $\mathbf{x}$.
Solution:

$$\mathbf{z} = \mathbf{U}^T \mathbf{x}$$

(ii) Write down the inverse formula to obtain the reconstructed signal in the right column in terms of linear transformation (change of basis) applied to the filtered spectrum $\hat{\mathbf{z}}$.
Solution:

$$\hat{\mathbf{x}} = \mathbf{U} \hat{\mathbf{z}}$$

(iii) What part of the signal would you discard to obtain the reconstructed signal? Draw a rectangle on each spectrum in the middle column where everything inside the rectangle is kept for the reconstruction.
Solution:



(iv) **False** The Fourier transform is a better choice than Wavelet for the first signal because the frequency components in the signal are global.

(v) **True** The Wavelet transform is a better choice than Fourier for the second signal because the signal has localized frequency components.

(vi) The first peaks in the spectrum correspond to the low-frequency components.

Problem 4 (Image compression):

Please find the solution in the iPython notebook `ex4-sol.ipynb` from

github.com/dalab/lecture_cil_public/tree/master/exercises/ex10/ex4-sol.ipynb,

and answer the questions.