Sparse Coding

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Overview

- Review: Orthogonality
- Fourier basis and Haar wavelets
- Matching pursuit
- Image processing

Orthogonality

Inner product

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\top \mathbf{v} = \sum_{i=1}^{d} \mathbf{u}_i \mathbf{v}_i,$$

Orthogonality

Two vectors $\mathbf{u}, \mathbf{v} \in H$ are orthogonal if and only if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Orthogonal matrix

Basis

A basis of a vector space is a set of vectors with the following two properties:

1. It is linearly independent
2. It spans the space

Orthogonal matrix

A basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is called orthonormal if

$$\mathbf{v}_i^\top \mathbf{v}_j = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}$$

A square matrix $\mathbf{A}$ with orthonormal columns is called an orthogonal matrix. The special case of $\mathbf{A}$ being an orthogonal matrix is important since the projection matrix becomes extremely simple since $\mathbf{A}^\top \mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

Projections and change of basis

Let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ be an orthonormal basis for $\mathbb{R}^n$.

Goal: write $\mathbf{x} \in \mathbb{R}^n$ as $\mathbf{x} = \sum_{i=1}^{n} a_i \mathbf{u}_i$ with real coefficients a_i.

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Observe that

$$
\begin{aligned}
\langle \mathbf{x}, \mathbf{u}_j \rangle &= \langle \sum_{i=1}^n a_i \mathbf{u}_i, \mathbf{u}_j \rangle \\
&= \sum_{\substack{i=1 \\ i \neq j}}^n a_i \langle \mathbf{u}_i, \mathbf{u}_j \rangle + a_j \langle \mathbf{u}_j, \mathbf{u}_j \rangle && \text{linearity} \\
&= a_j && \text{orthonormality}
\end{aligned}
$$

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This implies $\mathbf{x} = \sum_{i=1}^n \langle \mathbf{x}, \mathbf{u}_i \rangle \mathbf{u}_i$.

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With $\mathbf{U} = [\mathbf{u}_1 | \mathbf{u}_2 | \cdots | \mathbf{u}_n]$, the representation of $\mathbf{x}$ in terms of the new basis is $\mathbf{U}^\top \mathbf{x}$. Orthonormality is nice!

Energy Preservation

For an orthogonal matrix $\mathbf{U} \in \mathbb{R}^{n \times n}$ and vectors $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2 \tag{1}$$

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This implies that distances are preserved as well!

Pen&Paper: Problem 1, Series 9

Let $\mathbf{x} \in \mathbb{R}^K$ be a signal expressed in an orthonormal basis $\mathbf{U} \in \mathbb{R}^{k \times k}$ as:

$$\mathbf{x} = \sum_{k=1}^{K} z_k \mathbf{u}_k = \mathbf{U}\mathbf{z}$$

For a fixed basis, we want to find a good approximation $\hat{\mathbf{x}}$ for $\mathbf{x}$ using only $\tilde{K}$ coefficients ($\tilde{K} < K$) with a permutation of indices σ:

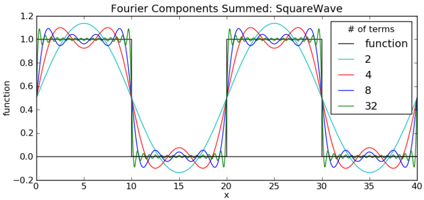
$$\hat{\mathbf{x}} = \sum_{k=1}^{\tilde{K}} z_{\sigma(k)} \mathbf{u}_{\sigma(k)}$$

Question: Find the permutation σ^{min} which minimizes the L^2 approximation error $\left\|\mathbf{x} - \hat{\mathbf{x}}_\sigma\right\|_2^2$:

$$\sigma^{min} = \underset{\sigma}{\operatorname{argmin}} \left\|\mathbf{x} - \hat{\mathbf{x}}_\sigma\right\|_2^2$$

Basis for functions

$$f(k) = \sum_n a_n \sin(\omega_n k)$$



DFT of a Signal

$$y = \sin(60 * 2\pi x) + 1.5\sin(80 * 2\pi x)$$

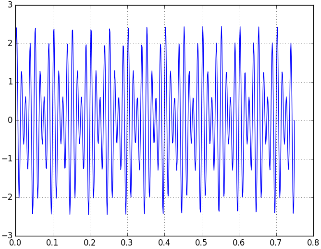


Figure: Original Signal

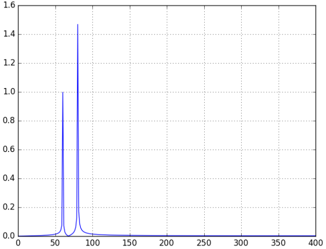


Figure: Fourier Transform

Build a different basis

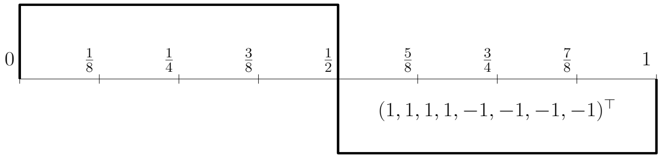
Fourier basis not sufficient for localized signals!

Build a different basis

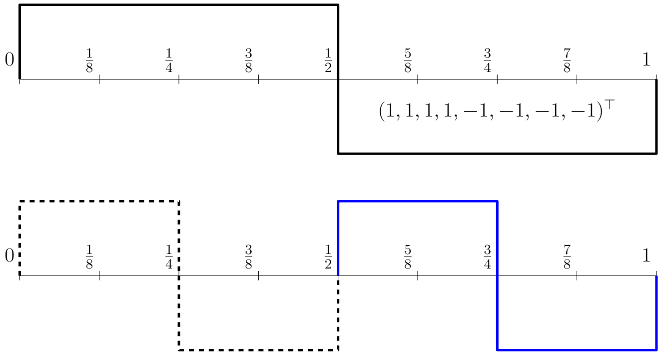
Fourier basis not sufficient for localized signals!

Want to build orthonormal basis for *nice* signals $[0, 1] \mapsto \mathbb{R}$.

Haar Wavelets



Haar Wavelets



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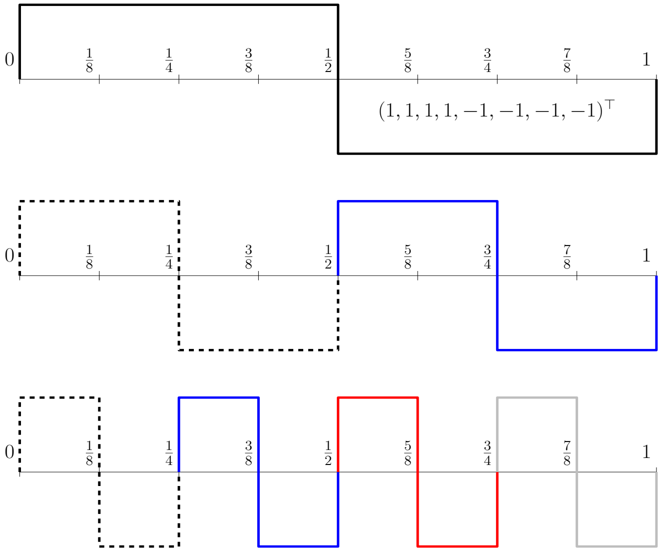

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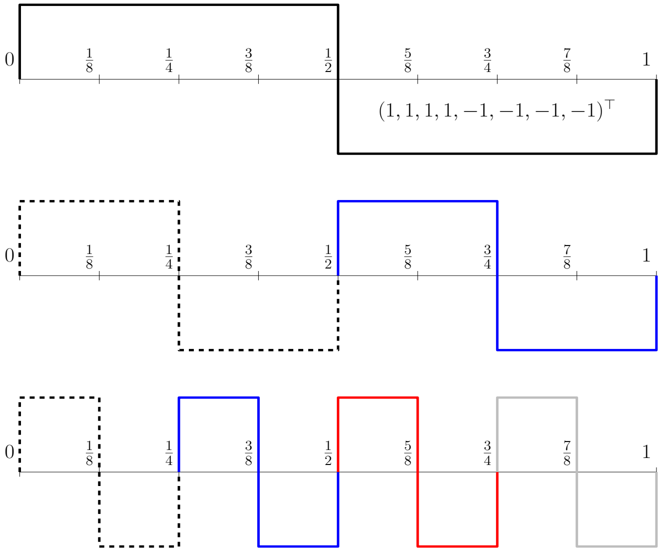

All functions have zero mean! Add $(1, 1, 1, 1, 1, 1, 1, 1)^\top$.

Haar wavelets matrix notation

Scale the vectors obtained from before to make basis orthonormal:

$$\mathbf{U} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & \sqrt{2} & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & \sqrt{2} & 0 & -2 & 0 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & -\sqrt{2} & 0 & 0 & -2 & 0 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & 2 \\ 1 & -1 & 0 & -\sqrt{2} & 0 & 0 & 0 & -2 \end{bmatrix}$$

$\mathbf{U}$ can be constructed recursively!

Haar Wavelets (Haar System on $[0, 1]$)

- Mother wavelet $\psi : [0, 1] \mapsto \mathbb{R}$,

$$\psi(t) = \begin{cases} 1 & \text{if } 0 \leq t < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

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  Use the mother wavelet to create the other Haar functions $\psi_{n,k} : [0, 1] \mapsto \mathbb{R}$, $\psi_{n,k}(t) = 2^{n/2}\psi(2^n t - k), \forall n, k \in \mathbb{N}_{\geq 0}$ such that $0 \leq k < 2^n$.

- Forms an orthonormal basis

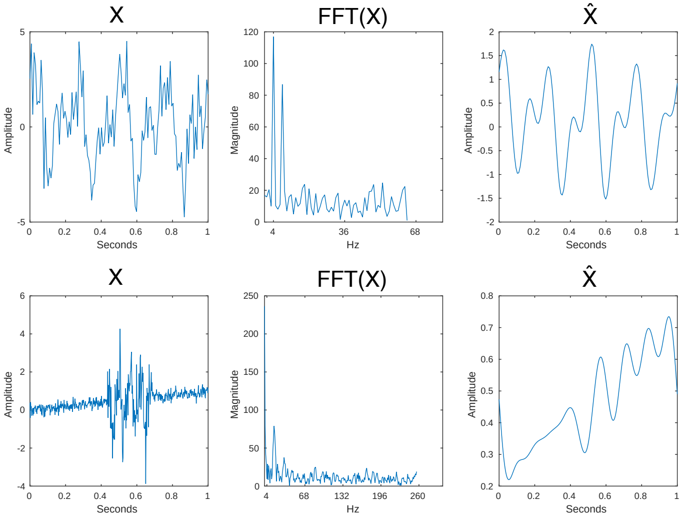
Notebook: Problem 2, Series 9

Please find the iPython notebook `ex2.ipynb` from

github.com/dalab/lecture_cil_public/tree/master/exercises/ex10/ex2.ipynb

and answer the questions.

Pen&Paper: Problem 3, Series 9



Overcomplete Dictionaries

Have a set of unit vectors (atoms) $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l$ that span $\mathbb{R}^n$ with $l > n$.
$\implies$ representation of $\mathbf{x} \in \mathbb{R}^n$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l$ is not unique!

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Goal: Want to find sparse representation of $\mathbf{x}$, i.e. find

$$\begin{aligned} \mathbf{z}^* \quad &\in \quad \arg\min_{\mathbf{z} \in \mathbb{R}^l} \|\mathbf{z}\|_0 \\ \text{s.t.} \quad & \qquad \mathbf{U}\mathbf{z} = \mathbf{x} \end{aligned}$$

where $\mathbf{U} = [\mathbf{u}_1 | \mathbf{u}_2 | \cdots | \mathbf{u}_n]$.
NP hard! :-(

Matching Pursuit

With initial residual $\mathbf{r}_0 = \mathbf{x}$ and initial approximation $\hat{\mathbf{x}}_0 = \mathbf{0}$, repeat:

- Find $j^* = \arg\max_j |\langle \mathbf{r}_i, \mathbf{u}_j \rangle|$
- Compute better approximation $\hat{\mathbf{x}}_{i+1} \leftarrow \hat{\mathbf{x}}_i + \langle \mathbf{r}_i, \mathbf{u}_{j^*} \rangle \mathbf{u}_{j^*}$
- Update residual $\mathbf{r}_{i+1} \leftarrow \mathbf{r}_i - \langle \mathbf{r}_i, \mathbf{u}_{j^*} \rangle \mathbf{u}_{j^*}$

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When should we stop?
Does it converge and if so, how fast?

Matching Pursuit Convergence

- If $\|\mathbf{r}_i\|_2^2$ converges to 0, then $\mathbf{r}_i$ converges to $\mathbf{0}$ and $\hat{\mathbf{x}}_i$ converges to $\mathbf{x}$

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- By the conservation of energy,

$$
\begin{aligned}
\|r_i\|_2^2 &= \langle \mathbf{r}_{i+1} + \langle \mathbf{r}_i, \mathbf{u}_{j^*} \rangle \mathbf{u}_{j^*}, \mathbf{r}_{i+1} + \langle \mathbf{r}_i, \mathbf{u}_{j^*} \rangle \mathbf{u}_{j^*} \rangle \\
&= \|\mathbf{r}_{i+1}\|_2^2 + 2\langle \mathbf{r}_{i+1}, \langle \mathbf{r}_i, \mathbf{u}_{j^*} \rangle \mathbf{u}_{j^*} \rangle + \|\langle \mathbf{r}_i, \mathbf{u}_{j^*} \rangle \mathbf{u}_{j^*}\|_2^2 && \text{linearity} \\
&= \|\mathbf{r}_{i+1}\|_2^2 + \|\langle \mathbf{r}_i, \mathbf{u}_{j^*} \rangle \mathbf{u}_{j^*}\|_2^2 && \perp \\
&= \|\mathbf{r}_{i+1}\|_2^2 + |\langle \mathbf{r}_i, \mathbf{u}_{j^*} \rangle|^2
\end{aligned}
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&= \|\mathbf{r}_{i+1}\|_2^2 + |\langle \mathbf{r}_i, \mathbf{u}_{j^*}\rangle|^2
\end{aligned}$$

- For $\|\mathbf{r}_i\|_2 \neq 0$, we have

$$\frac{\|r_{i+1}\|_2^2}{\|r_i\|_2^2} = 1 - \left|\left\langle \frac{\mathbf{r}_i}{\|\mathbf{r}_i\|_2}, \mathbf{u}_{j^*} \right\rangle\right|^2$$

- Want to bound $\left|\left\langle \frac{\mathbf{r}_i}{\|\mathbf{r}_i\|_2}, \mathbf{u}_{j^*} \right\rangle\right|^2$

Matching Pursuit Convergence

- With $\mathbf{v} \in \mathbb{R}^n$ s.t. $\|\mathbf{v}\|_2 = 1$,

$$\left|\left\langle \frac{\mathbf{r}_i}{\|\mathbf{r}_i\|_2}, \mathbf{u}_{j^*} \right\rangle\right| \geq \inf_{\mathbf{v}} \max_j |\langle \mathbf{v}, \mathbf{u}_j \rangle|$$

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- Idea: as $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l$ span $\mathbb{R}^n$, for $\mathbf{w} \in \mathbb{R}^n$, $\langle \mathbf{w}, \mathbf{u}_j \rangle = 0$ for all j if and only if $\mathbf{w} = \mathbf{0}$
- Therefore,

$$\left| \left\langle \frac{\mathbf{r}_i}{\|\mathbf{r}_i\|_2}, \mathbf{u}_{j^*} \right\rangle \right| > 0$$

Matching Pursuit Convergence Recap

$$\|\mathbf{r}_{i+1}\|_2^2 = 1 - \underbrace{\left\| \left\langle \frac{\mathbf{r}_i}{\|\mathbf{r}_i\|_2}, \mathbf{u}_{j^*} \right\rangle \right\|}_{0<\mu_{\min}\leq 1}{}^2 \|\mathbf{r}_i\|_2^2$$

There is some $\mu_{\min} \in]0, 1]$ s.t.

$$\|\mathbf{r}_i\|_2^2 \leq (1 - \mu_{\min}^2)^i \|\mathbf{r}_0\|_2^2$$

- This implies convergence!
- More details about MP in the following week.

Fourier Transform of an Image

How to take FT in 2-D?

- Image can be considered as a signal in 2D
- First take FT of the columns then FT of the rows (You can interchange them)

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How to interpret FT of an image?

- Large changes in the pixel values = High frequency
- Eg : edges, background objects

FT Example

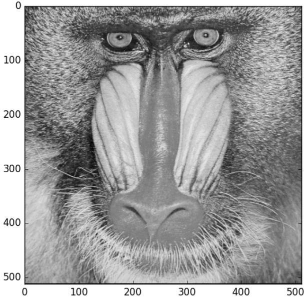

Figure: Original Image

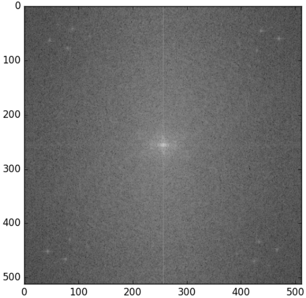

Figure: Frequency Spectrum

Image Compression by FT

- Reconstruct image by Inverse Fourier Transform using only the frequencies with largest magnitude

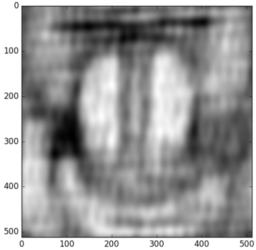

Figure: Using 0.1 percent

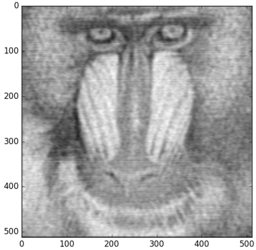

Figure: Using 1 percent

Discrete cosine transform

1D Discrete cosine transform:

$$z_k = \sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi}{N}\left(n + \frac{1}{2}\right) k\right] \qquad k = 0, \dots, N-1$$

2D Discrete cosine transform:

$$\begin{aligned} z_{k_1,k_2} &= \sum_{n_1=0}^{N_1-1}\left(\sum_{n_2=0}^{N_2-1} x_{n_1,n_2} \cos\left[\frac{\pi}{N_2}\left(n_2 + \frac{1}{2}\right) k_2\right]\right) \cos\left[\frac{\pi}{N_1}\left(n_1 + \frac{1}{2}\right) k_1\right] \\ &= \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x_{n_1,n_2} \cos\left[\frac{\pi}{N_1}\left(n_1 + \frac{1}{2}\right) k_1\right] \cos\left[\frac{\pi}{N_2}\left(n_2 + \frac{1}{2}\right) k_2\right]. \end{aligned}$$

Notebook: Problem 4, Series 9

Please find the iPython notebook `ex4.ipynb` from

github.com/dalab/lecture_cil_public/tree/master/exercises/ex10/ex4.ipynb

and answer the questions.